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1998 J. Phys. A: Math. Gen. 31 L177

(http://iopscience.iop.org/0305-4470/31/10/001)

J. Phys. A: Math. Gen. **31** (1998) L177–L183. Printed in the UK PII: S0305-4470(98)89241-X

**LETTER TO THE EDITOR**

# Exact zero-temperature critical behaviour of the ferromagnet in the uniaxial random field

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Received 14 November 1997, in final form 21 January 1998

**Abstract.** The zero-temperature critical behaviours of the classical Heisenberg and $XY$ ferromagnets in the uniaxial random field are studied in the arbitrary spatial dimension. Exact results are obtained for the magnetization, transversal to the random field direction, at the Gaussian and bimodal distributions of the random field. For the Gaussian distribution the critical behaviour in strong random fields is independent of the spatial dimension. The transversal magnetization, $m_\perp \sim \ln h_0/h_0^2$, where $h_0$ is the distribution width. For the bimodal distribution of the random field, the transversal magnetization obeys the law $m_\perp \sim \exp(-\text{constant}/(H_c - H)^{D/2})$, where $H_c$ is a critical field, and $H$ is the random field amplitude. The same critical behaviour is expected for related systems, for example random antiferromagnets in the uniform field.

## 1. Introduction

There has been an interest in random field magnets [1] for more than 20 years. This interest is motivated by the wide range of systems, which are described by random field models. Examples are dilute antiferromagnets in the magnetic field [2, 3], binary liquids in the porous media [4] and vortex phases of the doped superconductors [5]. A satisfactory theory of the random field magnets is still absent. Recently some progress was achieved due to the replica symmetry breaking variational method [6]. However, variational calculations are always approximate. This makes rigorous results for particular models important. Exact results [7] allowed us to solve the question of stability of long-range order in the random field ferromagnets. The problem of the critical behaviour is more difficult. At the moment it is solved only for the spherical models [8].

In this paper the critical behaviour of a random field model with a finite number of the order parameter components is exactly found in the arbitrary spatial dimension. We consider a multicomponent ferromagnet in the uniaxial random field at zero temperature. In the case of the bimodal distribution of the random field, the model describes low-temperature properties of the random antiferromagnet in the uniform field. We also consider the Gaussian distribution of the random field. The critical properties of this model at non-zero temperatures were studied in [9–11] with the renormalization group method. Here we investigate the zero-temperature critical behaviour of the magnetization, transversal to the random field. The critical behaviours are different for the Gaussian and bimodal cases and differ from what was found at non-zero temperatures. We neglect the quantum fluctuations assuming that the spins are large.

The model is described by the following Hamiltonian

$$H = -J \sum_{\langle ij\rangle} \boldsymbol{S}_i \boldsymbol{S}_j - \sum_i H_i S_i^z \tag{1}$$

where $\boldsymbol{S}_i$ are the unit length spin vectors, $H_i$ are the random fields and $\sum_{\langle ij\rangle}$ denotes the summation over the pairs of the neighbouring sites of the $D$-dimensional cubic lattice. The distance between the neighbouring spins will be designated as $a$. Two types of the field distribution are considered:

(1) the Gaussian distribution of the width $h_0$

$$P(H_i) = \frac{1}{\sqrt{2\pi}h_0}\exp\left(-\frac{H_i^2}{2h_0^2}\right) \tag{2}$$

(2) the bimodal distribution

$$P(H_i) = c\delta(H_i + H) + (1-c)\delta(H_i - H) \tag{3}$$

where $c$ and $1-c$ are the probabilities of the two directions of the random field $\pm H$. The Hamiltonian (1) with the bimodal field distribution can be obtained as a result of a gauge transformation of the Hamiltonian of the Mattis magnet [12] in the uniform field $H$. Results, found for the bimodal case, are also applicable to the random bond and random site antiferromagnets in the uniform field.

The transversal magnetization appears due to the same mechanism, as the transversal staggered magnetization in the antiferromagnet in the uniform field [13]. At the Gaussian field distribution, the transversal magnetization, $m_\perp$, is non-zero for any distribution width, $h_0 < \infty$. We shall show rigorously that at large $h_0$ the magnetization

$$m_\perp \sim \text{constant}\frac{\ln h_0}{h_0^2} \tag{4}$$

in any spatial dimension $D$. At the bimodal distribution, the transversal magnetization is zero for strong fields, $H > H_\mathrm{c} = 4DJ$. As we shall rigorously prove, at $H$ close to the critical field, $H_\mathrm{c}$, the magnetization

$$m_\perp \sim \exp\left(-\frac{\text{constant}}{(H_\mathrm{c} - H)^{D/2}}\right). \tag{5}$$

In the one-dimensional case the last formula is proven below for any strength of the disorder $c$ in equation (3). For $D > 1$ equation (5) is proven below at $c \ll 1$. Formula (5) is valid for an arbitrary strength of the disorder. A general proof will be published elsewhere.

The critical behaviour (4), (5) is the same for any number of the spin components. This is a manifestation of the fact, that in the ground state the transversal components of all the spins have the same direction. This fact can be verified by the consideration of how the energy (1) changes at spin rotations around the field direction. Thus, one can assume, that the spins $\boldsymbol{S}_i$ are two-component:

$$S_i^z = \cos\phi_i \qquad S_i^x = \sin\phi_i \geqslant 0. \tag{6}$$

The further investigation is simplified by the fact that the $x$-components of all the spins have the same sign. This circumstance reduces the problem to the calculation of the disorder average of the absolute value of any fixed spin:

$$|m_\perp| = \overline{|\sin\phi_i|}. \tag{7}$$

Below one can see the sketch of the derivations of equations (4) and (5).

## 2. The Gaussian distribution

We calculate the disorder average $\overline{\sin\phi_0}$ of the $x$-component of the spin $\boldsymbol{S}_0$ in some fixed site $S_0$ of the lattice. Our aim is to prove the inequalities

$$\overline{\sin\phi_0} > C_1 \frac{\ln h_0}{h_0^2} \tag{8a}$$

$$\overline{\sin\phi_0} < C_2 \frac{\ln h_0}{h_0^2} \tag{8b}$$

where $C_1, C_2$ are constants.

The origin of the lower estimation (8*a*) is seen from the two-spin model with the following Hamiltonian

$$H = -J\boldsymbol{S}_1\boldsymbol{S}_2 - H_1 S_1^z - H_2 S_2^z \tag{9}$$

where $H_1, H_2$ are random fields. Let the field $H_2 \gg J$ and the field $H_1$ be such that $H_1 + J \ll J^2/H_2$. Compare the energies of two equilibrium states of the system: the state $A$ with $S_1^x = S_2^x = 0$ and the state $B$ with $S_1^x \approx 1$, $S_2^x \approx J/H_2$. One easily sees that the state $B$ is deeper. Thus, the transversal magnetization is of order 1. For the Gaussian distribution of the fields $H_1, H_2$ with the width $h_0 \gg J$, the above configurations of the random fields have the probability $P \sim \ln h_0/h_0^2$. This provides the estimation (8*a*).

In the many-spin problem the main contribution to the transversal magnetization comes from the random field configurations with

$$|H_0 + J\sum_k \mathrm{sign} H_k| < \epsilon \sum_k \frac{J^2}{|H_k|} \tag{10}$$

$$|H_k| > \Omega J \qquad |H_p| > \Omega J \tag{11}$$

where $H_0$ is the field at the site $S_0$, $H_k$ are the fields at the neighbouring to $S_0$ sites, $H_p$ the fields at the neighbours of the nearest neighbours of $S_0$, the constants $\epsilon \ll 1$, $\Omega \gg 1$. The probability of the configuration (10), (11) is of order $\ln h_0/h_0^2$. Hence, to prove the inequality (8*a*) it suffices to show that $\sin\phi_0 \sim 1$ under the conditions (10), (11). To make this, we compare the energy of the lowest state with $\sin\phi_0 < \epsilon$, $A$, and the energy of the described below state $B$ with $\sin\phi_0 = 1$. In the state $B$, all the spins, except $\boldsymbol{S}_0$ and its neighbours, are chosen to have the same direction, as in the state $A$, and the neighbours, $\boldsymbol{S}_k$, correspond to the minimum of the energy:

$$\sin\phi_k = \cos\phi_k \frac{J\sum_l \sin\phi_l}{H_k + J\sum_l \cos\phi_l} \tag{12}$$

where $\sum_l$ denotes the summation over all the neighbours of the spin $\boldsymbol{S}_k$. Analogously to the two-spin example, the state $B$ is deeper than $A$. This allows us to prove equation (8*a*).

The same arguments show that the transversal magnetization is non-zero for any $h_0 < \infty$.

Let us derive equation (8*b*). For the strong random field $H_0$ at the site $S_0$, the spin $\boldsymbol{S}_0$ is oriented almost along the field. Only spins in the weak fields provide contributions to the transversal magnetization. These contributions depend on the random fields at the neighbouring sites. However, they are independent of the random fields at the distant spins, since the correlations between the transversal components of the spins rapidly decrease when the distance is increased. It turns out that for large $h_0$ the magnetization is sensitive to the values of the random fields only within the cube $\Gamma$ with the edge length equal to $9a$

and with the centre in the site $S_0$ ($a$ is the distance between the neighbouring spins). This allows us to reduce the investigation to the consideration of the following four possibilities.

(1) At least in two points in the cube $\Gamma$ the random fields $H_i < \Omega J$, where the constant $\Omega \gg 1$.

(2) In all the points of $\Gamma$ $H_i > \Omega J$.

(3) The field $H_i < \Omega J$ in one non-central point of the cube.

(4) The field $H_0 < \Omega J$.

The probability of the first case is of order $1/h_0^2$. Hence, the corresponding random field realizations provide the contribution, $m_1$, to the transversal magnetization

$$m_1 < \frac{\text{constant}}{h_0^2}. \tag{13}$$

In the second case the spin $\boldsymbol{S}_0$ behaves almost as in the strong uniform field, and the transversal magnetization is small. One can estimate the magnetization, using equation (12) three times, estimating $\sin\phi_k$ and $\cos\phi_k$ in the right-hand side of the final formula with their extremal values, and calculating the disorder average. The resulting contribution to $m_\perp$

$$m_2 < \text{constant}\left(\frac{\ln h_0}{h_0}\right)^3. \tag{14}$$

In the third case, there are two possibilities:

(3a) in the cube $\Delta$ with the edge length equal to $5a$ and with the centre in the site $S_0$, the random field $H_i > \Omega J$;

(3b) there is a point $S_1$ in the cube $\Delta$, where the field $H_1 < \Omega J$.

Case (3a) can be studied by the same method, as the case (2). The resulting contribution to the transversal magnetization

$$m_{3\text{a}} < \text{constant}\left(\frac{\ln h_0}{h_0}\right)^3. \tag{15}$$

In case (3b) equation (12) provides the following estimation of the contribution to $m_\perp$:

$$m_{3\text{b}} < \text{constant}\left(\frac{\ln h_0}{h_0}\right)^3 + \text{constant}\ (p\overline{\sin\phi_1}) \tag{16}$$

where $\sin\phi_1$ is the $x$-component of the spin at the site $S_1$, $p$ is the probability of the configuration (3b). The average, $\overline{\sin\phi_1}$, is estimated analogously to the consideration of case (4).

In the fourth case there are two variants:

(4a) $|H_0 + J\sum_k \text{sign}\ H_k| < C\sum_k \frac{J^2}{|H_k|}$;

(4b) $|H_0 + J\sum_k \text{sign}\ H_k| > C\sum_k \frac{J^2}{|H_k|}$,

where the notations are the same as in equation (10), the constant $C \gg 1$. The probability of configuration (4a) is of order $\ln h_0/h_0^2$. Hence, the contribution of this configuration to the transversal magnetization

$$m_{4\text{a}} < \text{constant}\frac{\ln h_0}{h_0^2}. \tag{17}$$

In case (4b) $\sin\phi_0 \ll 1$. Calculations with equation (12) provide the estimation

$$m_{4\text{b}} < \text{constant}\left(\frac{\ln h_0}{h_0}\right)^3. \tag{18}$$

Combining inequalities (13)–(18), one obtains equation (8$b$). This proves equation (4).

## 3. The bimodal distribution

In this case, the main contribution to the magnetization comes from the regions with the random fields oriented up and down in the chess order. We shall call the regions, in any point of which the direction of the field is opposite to the direction of the random fields in all the neighbouring sites, as the chess regions. In the chess regions, the Hamiltonian (1) can be obtained from the Hamiltonian of the antiferromagnet in the uniform field by the inversion, $\boldsymbol{S}_i \to -\boldsymbol{S}_i$, of the spins of one of the two 'chess' sublattices. This implies, that in the large chess regions, the transversal magnetization appears at the same critical field, as in the 'clean' antiferromagnet. In our model this field, $H_c = 4DJ$. One can check, that $m_\perp \neq 0$ for any $H < H_c$.

The critical behaviour (5) is a consequence of the fact that the large chess regions are exponentially rare. More exactly, the large chess regions are exponentially rare at the weak disorder ($c \ll 1$ or $1 - c \ll 1$) in the spatial dimensions $D > 1$, and for any strength of the disorder at $D = 1$. The question of what happens above the percolation threshold is beyond the scope of this letter.

Our aim is to prove the inequalities

$$m_\perp > \exp\left(-\frac{C_1}{(H_c - H)^{D/2}}\right) \tag{19$a$}$$

$$m_\perp < \exp\left(-\frac{C_2}{(H_c - H)^{D/2}}\right) \tag{19$b$}$$

where $C_1, C_2$ are constants. Equation (19$b$) is proven below for $D \neq 2$. In the two-dimensional case we shall prove a weaker inequality:

$$m_\perp < \exp\left(-\frac{\text{constant}}{(H_c - H)\ln(1/(H_c - H))}\right). \tag{19$c$}$$

It is interesting to elucidate whether logarithmical corrections are present in the two-dimensional system, or equation (19$b$) is always valid.

To deduce equation (19$a$), consider a chess region of the size $L \gg a$. Such regions have concentrations of order $\exp(-\text{constant } L^D)$. For sufficiently large $L$, the magnetization in the centre of the region is the same, as in the pure antiferromagnet, $\sin\phi \sim \sqrt{1 - H/H_c}$. Comparison of the different contributions to the energy shows that the state with $\sin\phi \sim \sqrt{1 - H/H_c}$ becomes favourable at $L \sim 1/\sqrt{1 - H/H_c}$. This allows us to prove equation (19$a$).

Outside the chess regions the transversal magnetization is a rapidly decreasing function of the distance from the nearest chess region. One can estimate the rate of decrease with equation (12). One rewrites (12) outside the chess region at $H \approx 4DJ$ in the following form

$$\sin\phi_k < \frac{J\sum_l \sin\phi_l}{H - (2D - 1)J}. \tag{20}$$

Applying equation (20) several times, one obtains for the transversal component of the spin $\boldsymbol{S}_k$

$$\sin\phi_k < \left[\frac{2DJ}{H - (2D - 1)J}\right]^S \tag{21}$$

where $S$ is the distance from the spin $\boldsymbol{S}_k$ to the nearest chess region. To deduce equation (19$b$), one needs an estimation of the same structure, as (21), but with a larger exponent than $S$.

To find such an estimation, consider a small chess region of the volume $V \ll V_c$, where

$$V_c \sim (H_c - H)^{-D/2} \qquad D \neq 2 \tag{22a}$$

$$V_c \sim \frac{1}{(H_c - H)\ln(1/(H_c - H))} \qquad D = 2. \tag{22b}$$

Inside the region, equation (12) implies the following inequality:

$$\Delta \sin\phi \geqslant -\frac{H_c - H}{J}(\sin\phi)_{\max} \tag{23}$$

where

$$\Delta \sin\phi = \sum_{\text{neighbours}} \sin\phi_k - 2D\sin\phi \tag{24}$$

is the lattice Laplacian, $(\sin\phi)_{\max}$ the maximal value of the transversal magnetization in the region. One can represent a solution of the Poisson equation as the sum of a particular solution and a solution of the Laplace equation. The particular solution can be chosen in the form of the potential of the mass distribution from the right-hand side of the equation. The solution of the Laplace equation can be estimated with the principal of maximum. This allows us to obtain from equation (23) that

$$\sin\phi < (1 + \epsilon)(\sin\phi_b)_{\max} \tag{25}$$

where $\epsilon \ll 1$, $(\sin\phi_b)_{\max}$ is the maximal magnetization on the border of the region. Equation (25) provides an estimation of the magnetization in the chess region via the magnetization outside the region. With equations (20) and (25) one finds that

$$\sin\phi_k < \text{constant}\left[\frac{2DJ(1+\epsilon)}{H - (2D-1)J}\right]^{S'/V_c} \tag{26}$$

where $V_c$ is a critical volume (22), $S'$ is the distance from the spin $\boldsymbol{S}_k$ to the nearest chess region of the volume $V > V_c$. Since such regions are exponentially rare, equation (26) allows us to obtain equations (19*b*) and (19*c*).

## 4. Discussion

The ordering in the studied system is determined by the rare regions. This resembles the Griffiths phase of the impure magnets [14]. However, in contrast to the Griffiths transition, in our problem there is a spontaneous symmetry breaking and a long-range order. The order appears due to a weak interaction of the ordered regions. Such interaction exists also in other disordered systems. However, it does not necessarily lead to the appearance of the long-range order at non-zero temperatures. Near the Griffiths point this interaction is weak and thermal fluctuations destroy the order. This happens for example in the random bond ferromagnet. In our problem there are no thermal fluctuations and thus the system is always in the ground state. Below the critical field at which rare ordered regions appear, this state is ferromagnetical.

Since the mean field approximation [9] ignores contributions from the rare regions, it provides wrong results. For the Gaussian random field distribution, the mean field theory predicts an incorrect exponential dependence of the magnetization on the distribution width. In the bimodal case, this approximation gives both an incorrect critical field and an incorrect critical behaviour.

Results, obtained for the bimodal distribution, are applicable for many related systems. In particular, the critical behaviour (5) is expected for the random antiferromagnets in the uniform field.

The considered model does not include the quantum effects. Strong quantum fluctuations can completely transform behaviour of the system. If the quantum fluctuations are weak, they are relevant only near the critical field. Thus, for the weak quantum fluctuations there are two regimes in the critical domain: fast magnetization decreasing (5) as the field increases (in weaker fields), and slow decreasing in the quantum regime (in stronger fields). As a result, the graph of the magnetization contains a step. I believe, that qualitatively the same behaviour can be found for the random antiferromagnets in the uniform field even for not very weak fluctuations. It would be interesting to observe such a behaviour experimentally.

I thank V S Dotsenko for useful discussions. This work was supported by RFBR grant No 96-02-18985.

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